Malaysian Journal of Mathematical Sciences 15(3): 347–356 (2021)

Malaysian Journal of Mathematical Sciences

Journal homepage: https://einspem.upm.edu.my/journal

Existence and Uniqueness of Solutions for a Nonlinear Fractional Elliptic System

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Received: 21 September 2020
Accepted: 26 April 2021

Abstract

In this article, we study the existence and uniqueness of weak solution for the non-linear fractional elliptic system

$$\begin{cases} (-\Delta)^s\varphi(z) = p(z,\varphi(z),\phi(z)) & \text{in} \quad \Omega, \\ (-\Delta)^s\phi(z) = k(z,\varphi(z),\phi(z)) & \text{in} \quad \Omega, \\ \varphi = \phi = 0 & \text{on} \quad \mathbb{R}^n \setminus \Omega, \end{cases}$$

with $s \in (0,1)$ and Ω is an open bounded subset of $\mathbb{R}^n$. We use the Schauder fixed point theorem to prove the existence of solution under suitable assumptions on the nonlinearities p and k, and the contraction principle to prove the existence and uniqueness of solution in a particular case.

Keywords: nonlinear elliptic equations; fractional Laplacian; weak solution.

1 Introduction

Fractional differential equations involve derivatives of fractional order are important mathematical models of some functional ways to some of the problems in several disciplines like in image denoising, natural sciences and different other branches. As a result, the question of fractional differential equations is attracting a lot of attention, for example see the lecture notes [5] which are devoted to the analysis of a nonlocal equation in the whole of Euclidean space, the authors in the paper [11] and the references therein are concerned with the existence of solutions for the fractional problems.

Another aspect in the study of coupled systems is when involving fractional differential equations seems crucial as such systems occur in diverse problems of applied sciences. See the article [12] which prove the existence of solutions for a type of fractional systems.

This work is devoted to the study of the existence of solution to a system of nonlocol equations involving the fractional Laplacian. These equations have a variational structure, we find nontrivial solution for them.

The Dirichlet problem for the fractional Laplacian has been studied from the point of view of probability, potential theory, and PDEs. It has attracted lots of interest, as his spectrum in [8] and his definition from a standpoint of probability in [2].

The fractional Laplacian $(-\Delta)^s$ is defined as follows:

$$(-\Delta)^s\psi(z) = K(n,s)P.V\int_{\mathbb{R}^n}\frac{\psi(z)-\psi(w)}{|z-w|^{n+2s}}dw,$$

along $\varphi \in C_0^\infty(\mathbb{R}^n)$, where $s \in (0,1)$, $P.V$; denotes the integral in the sense of the principal value and

$$K(n,s) = \frac{4^s\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}}\frac{s}{\Gamma(1-s)}.$$

In this paper, we adopt a fixed point theorem in order to confirm the existence of a weak solutions to the system

$$\begin{cases}(-\Delta)^s\varphi(z) = p(z,\varphi(z),\phi(z)) & \text{in} \quad \Omega,\\ (-\Delta)^s\phi(z) = k(z,\varphi(z),\phi(z)) & \text{in} \quad \Omega,\\ \varphi = \phi = 0 & \text{on} \quad \mathbb{R}^n\setminus\Omega,\end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, $s \in (0,1)$ and $p,k : \Omega\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are two continuous functions satisfying the Carathéodory conditions (i. e. : $p(.,x)$, $k(.,y)$) are measurables for each $x,y \in \mathbb{R}^2$ and $p(z,.)$, $k(w,.)$ are continuous for almost every $z,w \in \Omega$, and also verifying the growth restriction defined below

$$\begin{cases}|p(z,\xi_1,\xi_2)| \le r_1(z) + a|\xi_1|^{\delta_1} + b|\xi_2|^{\delta_1},\\ |k(z,\eta_1,\eta_2)| \le r_2(z) + c|\eta_1|^{\delta_2} + d|\eta_2|^{\delta_2}.\end{cases} \tag{2}$$

We employed the notation $|.|$ that stands for absolute value in $\mathbb{R}$, the constants δ_1 and δ_2 are in the closed interval $[0,1]$ and $r = (r_1,r_2) \in (L^2(\Omega))^2$ nonnul function; a, b, c and d are nonnegative constants.

As far as we know, this result is new and represent fractional version of the classical theorem see [6]. We also need to mention that the linear case has already been studied in a lot of works.

The remaining parts in this paper are as follows. Section 2 is an introduction to basic definitions as well as the main result of this paper. In Section 3, a fixed point formulation of the system (1) is presented, and in Section 4, we prove the main result. Finally, in Section 5, we give a particule case.

2 Preliminaries and The Main Result

We consider the space

$$U = E^{s,2}(\Omega) \times E^{s,2}(\Omega), \tag{3}$$

with the norm

$$\|(\varphi, \phi)\|_U^2 = \|\varphi\|^2_{E^{s,2}(\Omega)} + \|\phi\|^2_{E^{s,2}(\Omega)},$$

where $E^{s,2}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\|_{H^s}}$ ($E^{s,2}(\Omega)$ is the completion of $C_c^\infty(\Omega)$ compared to the $H^s(\Omega)$ norm), if Ω is a bounded Lipschitz open set, then

$$E^{s,2}(\Omega) = \{\varphi \in H^s(\mathbb{R}^n), \text{ such that } \varphi = 0 \text{ in } \mathbb{R}^n \setminus \Omega\},$$

such that

$$H^s(\mathbb{R}^n) = \{\varphi \in L^2(\mathbb{R}^n) : \frac{|\varphi(z) - \varphi(w)|}{|z-w|^{\frac{n}{2}+s}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n)\}.$$

Thus, $E^{s,2}(\Omega)$ is a Hilbert space with respect to the scalar product

$$\langle \varphi, \phi \rangle = K(n,s) \iint_{\mathbb{R}^{2n}} \frac{(\varphi(z) - \varphi(w))(\phi(z) - \phi(w))}{|z-w|^{n+2s}} dwdz.$$

The norm in $E^{s,2}(\Omega)$ is,

$$\|\varphi\|_{E^{s,2}(\Omega)} = \left[\iint_{\mathbb{R}^{2n}} \frac{|\varphi(z) - \varphi(w)|^2}{|z-w|^{n+2s}} dwdz\right]^{\frac{1}{2}}.$$

Proposition 2.1. *(see [9] and [14]) Let Ω be a bounded Lipschitzian subset of $\mathbb{R}^n$ and $s \in (0,1)$ such that $n > 2s$. Let ψ be a measurable function compactly supported defined from Ω to $\mathbb{R}$. Then, there exists a positive constant $c_{emb} > 0$ (embedding constant) depending on n and s such that,*

$$\|\psi\|_{L^2(\Omega)} \leq c_{emb} \|\psi\|_{E^{s,2}(\Omega)}.$$

Proposition 2.2. *(see [10]) Let $s \in (0,1)$, $n \geq 1$, $\Omega \in \mathbb{R}^n$ be a Lipschitz bounded open set and $\Im$ be a bounded subset of $L^2(\Omega)$. Suppose that,*

$$\sup_{\psi \in \Im} \int_\Omega \int_\Omega \frac{|\psi(z) - \psi(w)|^2}{|z-w|^{n+2s}} dzdw < +\infty,$$

then $\Im$ is precompact in $L^2(\Omega)$.

All along the paper and without further mention, we always assume that $n > 2s$ (condition of the continuous embedding) and $\|.\|_{L^2(\Omega)}$ will denote the usual norm on $L^2(\Omega)$. We set

$$V = L^2(\Omega) \times L^2(\Omega). \tag{4}$$

Theorem 2.1. *(Schauder fixed point, see [7]) Let F be a Banach space, $R > 0$, $B_R = \{z \in F, \|z\| \leq R\}$ and ψ a compact application from B_R to B_R (that is, ψ continuous and $\{\psi(z), z \in B_R\}$ relatively compact in F). Then ψ admits a fixed point, that is, there is $z \in B_R$ such that $\psi(z) = z$.*

The main result of this paper is

Theorem 2.2. *System (1) has at least one solution $(\varphi, \phi) \in U$, if hypothesis (2) is fulfilled.*

3 Fixed Point Formulation

From the definition of the fractional Laplacian $(-\Delta)^s$, the system (1) is weakly formulated as follows:

$$\begin{cases} K(n,s) \iint_{\mathbb{R}^{2n}} \frac{(\varphi(z)-\varphi(w))(\mu(z)-\mu(w))}{|z-w|^{n+2s}} dwdz = \int_\Omega p(z,\varphi(z),\phi(z))\mu(z)dz, \\ K(n,s) \iint_{\mathbb{R}^{2n}} \frac{(\phi(z)-\phi(w))(\nu(z)-\nu(w))}{|z-w|^{n+2s}} dwdz = \int_\Omega k(z,\varphi(z),\phi(z))\nu(z)dz, \end{cases}$$

for $(\mu, \nu) \in U$. We define

$$\hat{M}_{\varphi,\phi} : (\mu,\nu) \mapsto (\hat{M}_\varphi(\mu), \hat{M}_\phi(\nu)),$$
$$\hat{N}_{\varphi,\phi} : (\mu,\nu) \mapsto (\hat{N}_1(\mu), \hat{N}_2(\nu)),$$

where

$$\hat{M}_\varphi(\mu) = K(n,s) \iint_{\mathbb{R}^{2n}} \frac{(\varphi(z)-\varphi(w))(\mu(z)-\mu(w))}{|z-w|^{n+2s}} dwdz,$$
$$\hat{M}_\phi(\nu) = K(n,s) \iint_{\mathbb{R}^{2n}} \frac{(\phi(z)-\phi(w))(\nu(z)-\nu(w))}{|z-w|^{n+2s}} dwdz,$$

and

$$\hat{N}_1(\mu) = \int_\Omega p(z,\varphi(z),\phi(z))\mu(z)dz,$$
$$\hat{N}_2(\nu) = \int_\Omega k(z,\varphi(z),\phi(z))\nu(z)dz.$$

Lemma 3.1. *On the functional space U the operators $\hat{M}$ and $\hat{N}$ are linear and continuous.*

Since U is a Hilbert space, we can use the Riesz representation theorem (see [6] Theorem 1.2.40) to prove that there exists uniquely determined elements $M(\varphi, \phi)$, $N(\varphi, \phi) \in U$ such that

$$M(\varphi,\phi) = (M(\varphi), M(\phi)) \text{ and } N(\varphi,\phi) = (N_1(\varphi,\phi), N_2(\varphi,\phi)).$$

We have also

$$\begin{cases} \hat{M}_\varphi(\mu) = \langle \hat{M}_\varphi, \mu \rangle_{\langle (E^{s,2})', E^{s,2} \rangle} = \langle M(\varphi), \mu \rangle_{\langle E^{s,2}, E^{s,2} \rangle}, \\ \hat{M}_\phi(\nu) = \langle \hat{M}_\phi, \nu \rangle_{\langle (E^{s,2})', E^{s,2} \rangle} = \langle M(\phi), \nu \rangle_{\langle E^{s,2}, E^{s,2} \rangle}, \end{cases}$$

and

$$\begin{cases} \hat{N}_1(\mu) = \langle \hat{N}_1, \mu \rangle_{\langle (E^{s,2})', E^{s,2} \rangle} = \langle N_1(\varphi, \phi), \mu \rangle_{\langle E^{s,2}, E^{s,2} \rangle}, \\ \hat{N}_2(\nu) = \langle \hat{N}_2, \nu \rangle_{\langle (E^{s,2})', E^{s,2} \rangle} = \langle N_2(\varphi, \phi), \nu \rangle_{\langle E^{s,2}, E^{s,2} \rangle}, \end{cases}$$

for all $(\mu, \nu) \in U$.

The standard norms of $M(\varphi, \phi)$ and $N(\varphi, \phi)$ are defined by:

$$\begin{cases} \|M(\varphi, \phi)\|_U^2 = \|M(\varphi)\|_{E^{s,2}}^2 + \|M(\phi)\|_{E^{s,2}}^2, \\ \|N(\varphi, \phi)\|_U^2 = \|N_1(\varphi, \phi)\|_{E^{s,2}}^2 + \|N_2(\varphi, \phi)\|_{E^{s,2}}^2, \end{cases}$$

where

$$\begin{cases} \|M(\varphi)\|_{E^{s,2}} = \|\hat{M}_\varphi\|_{(E^{s,2})'} = \sup_{\|\mu\| \leq 1} |\langle M\varphi, \mu \rangle|, \\ \|M(\phi)\|_{E^{s,2}} = \|\hat{M}_\phi\|_{(E^{s,2})'} = \sup_{\|\nu\| \leq 1} |\langle M\phi, \nu \rangle|, \end{cases}$$

and

$$\begin{cases} \|N_1(\varphi, \phi)\|_{E^{s,2}} = \|\hat{N}_1\|_{(E^{s,2})'} = \sup_{\|\mu\| \leq 1} |\langle N_1(\varphi, \phi), \mu \rangle|, \\ \|N_2(\varphi, \phi)\|_{E^{s,2}} = \|\hat{N}_2\|_{(E^{s,2})'} = \sup_{\|\nu\| \leq 1} |\langle N_2(\varphi, \phi), \nu \rangle|. \end{cases}$$

To demonstrate that the Dirichlet problem (1) has at least one weak solution, it is necessary and sufficient to prove that the operator equation

$$M(\varphi, \phi) = N(\varphi, \phi), \tag{5}$$

has at least one solution in the space U.

There are different equivalent sub-products defined on $E^{s,2}(\Omega)$. If we choose

$$\langle \varphi, \phi \rangle = K(n, s) \iint_{\mathbb{R}^{2n}} \frac{(\varphi(z) - \varphi(w))(\phi(z) - \phi(w))}{|z - w|^{n+2s}} dwdz,$$

then M defined by (5) is just an identity on U.
Hence (5) is equivalent in U to

$$(\varphi, \phi) = N(\varphi, \phi). \tag{6}$$

4 Main Result

In this section, we will present different lemmas to arrive at the existence of a weak solution of system (1). Our methods of proof is based on the application of the Schauder fixed point theorem.

Lemma 4.1. *The operator N is continuous in U.*

Proof. Let $(\varphi_n, \phi_n) \longrightarrow (\varphi, \phi)$ in U, then we have,

$$\begin{cases} \|N_1(\varphi_n, \phi_n) - N_1(\varphi, \phi)\|_{E^{s,2}} = \sup_{\|\mu\| \leq 1} |\langle N_1(\varphi_n, \phi_n) - N_1(\varphi, \phi), \mu \rangle|, \\ \|N_2(\varphi_n, \phi_n) - N_2(\varphi, \phi)\|_{E^{s,2}} = \sup_{\|\nu\| \leq 1} |\langle N_2(\varphi_n, \phi_n) - N_2(\varphi, \phi), \nu \rangle|. \end{cases}$$

Since

$$\begin{cases} \sup_{\|\mu\|\leq 1}|\langle N_1(\varphi_n,\phi_n)-N_1(\varphi,\phi),\mu\rangle| \leq c_{emb}\|p(z,\varphi_n,\phi_n)-p(z,\varphi,\phi)\|_{L^2}, \\ \sup_{\|\nu\|\leq 1}|\langle N_2(\varphi_n,\phi_n)-N_2(\varphi,\phi),\nu\rangle| \leq c_{emb}\|k(z,\varphi_n,\phi_n)-k(z,\varphi,\phi)\|_{L^2}, \end{cases}$$

thus,

$$\|N(\varphi_n,\phi_n)-N(\varphi,\phi)\|_U^2 \leq c_{emb}^2\|p(z,\varphi_n,\phi_n)-p(z,\varphi,\phi)\|_{L^2}^2 + c_{emb}^2\|k(z,\varphi_n,\phi_n)-k(z,\varphi,\phi)\|_{L^2}^2.$$

When $n\to\infty$, the right-hand side approaches zero it follows from the continuity of the Nemytski operators from $L^2(\Omega)$ in $L^2(\Omega)$ (we have $E^{s,2}(\Omega)\subset L^2(\Omega)$). This proves the continuity of N (we put that $(\varphi_n,\phi_n)\longrightarrow(\varphi,\phi)$ and we arrived at $N(\varphi_n,\phi_n)\longrightarrow N(\varphi,\phi)$ which proves the continuity of N). □

Lemma 4.2. *The operator N is compact.*

Proof. Let $S\subset U$ be a bounded set and $\{w_n\}_{n=1}^\infty=\{w_{1,n},w_{2,n}\}_{n=1}^\infty\subset N(S)$ be an arbitrary sequence. Let $\{\varphi_n,\phi_n\}_{n=1}^\infty\subset S$ be such that

$$N(\varphi_n,\phi_n)=(w_{1n},w_{2n}).$$

The reflexivity of U implies that $(\varphi_n,\phi_n)\rightharpoonup(\varphi,\phi)$ in U at least for a subsequence. As a result of the compact injection of $E^{s,2}(\Omega)$ in $L^2(\Omega)$ (Proposition 2.2) that $(\varphi_n,\phi_n)\to(\varphi,\phi)$ in V (the space V is defined by (4)). Ideas similar to those in the proof of Lemma 4.1 yield,

$$(w_{1n},w_{2n})\longrightarrow N(\varphi,\phi),\ \text{in } U$$

(at least for a subsequence). This proves the compactness of $\overline{N(S)}$, i.e., N is a compact operator. □

Lemma 4.3. *The operator N maps the closure of the ball $B(0;R)\subset U$ into itself.*

Proof. For all $(\varphi,\phi)\in U$, we have from Section 3,

$$\begin{cases} \|N_1(\varphi,\phi)\|_{E^{s,2}} = \sup_{\|\mu\|\leq 1}|\langle N_1(\varphi,\phi),\mu\rangle|, \\ \|N_2(\varphi,\phi)\|_{E^{s,2}} = \sup_{\|\nu\|\leq 1}|\langle N_2(\varphi,\phi),\nu\rangle|. \end{cases}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{cases} \|N_1(\varphi,\phi)\|_{E^{s,2}} \leq c_{emb}(\int_\Omega|p(z,\varphi(z),\phi(z))|^2dz)^{\frac{1}{2}}, \\ \|N_2(\varphi,\phi)\|_{E^{s,2}} \leq c_{emb}(\int_\Omega|k(z,\varphi(z),\phi(z))|^2dz)^{\frac{1}{2}}. \end{cases}$$

From the hypothesis (2), we get

$$\begin{cases} \|N_1(\varphi,\phi)\|_{E^{s,2}} \leq c_{emb}(\int_\Omega|r_1(z)+a|\varphi(z)|^{\delta_1}+b|\phi(z)|^{\delta_1}|^2dz)^{\frac{1}{2}}, \\ \|N_2(\varphi,\phi)\|_{E^{s,2}} \leq c_{emb}(\int_\Omega|r_2(z)+c|\varphi(z)|^{\delta_2}+d|\phi(z)|^{\delta_2}|^2dz)^{\frac{1}{2}}, \end{cases}$$

and
$$\begin{cases} \|N_1(\varphi,\phi)\|_{E^{s,2}} \leq c_{emb}\big(\|r_1\|_{L^2} + a(\int_\Omega |\varphi(z)|^{2\delta_1}dz)^{\frac{1}{2}} + b(\int_\Omega |\phi(z)|^{2\delta_1}dz)^{\frac{1}{2}}\big), \\ \|N_2(\varphi,\phi)\|_{E^{s,2}} \leq c_{emb}\big(\|r_2\|_{L^2} + c(\int_\Omega |\varphi(z)|^{2\delta_2}dz)^{\frac{1}{2}} + d(\int_\Omega |\phi(z)|^{2\delta_2}dz)^{\frac{1}{2}}\big), \end{cases} \tag{7}$$

where the last estimate (7) is due to the Minkowski inequality for $p = 2$. Applying the Hölder inequality, we have

$$\begin{cases} (\int_\Omega|\varphi(z)|^{2\delta_1}dz)^{\frac{1}{2}} \leq (\int_\Omega|\varphi(z)|^{2}dz)^{\frac{\delta_1}{2}}(mes(\Omega))^{\frac{1-\delta_1}{2}}, \\ (\int_\Omega|\phi(z)|^{2\delta_2}dz)^{\frac{1}{2}} \leq (\int_\Omega|\phi(z)|^{2}dz)^{\frac{\delta_2}{2}}(mes(\Omega))^{\frac{1-\delta_2}{2}}, \end{cases}$$

then,
$$\begin{cases} (\int_\Omega|\varphi(z)|^{2\delta_1}dz)^{\frac{1}{2}} \leq c_{emb}^{\delta_1}(mes(\Omega))^{\frac{1-\delta_1}{2}}\|\varphi\|^{\delta_1}_{E^{s,2}(\Omega)}, \\ (\int_\Omega|\phi(z)|^{2\delta_2}dz)^{\frac{1}{2}} \leq c_{emb}^{\delta_2}(mes(\Omega))^{\frac{1-\delta_2}{2}}\|\phi\|^{\delta_2}_{E^{s,2}(\Omega)}. \end{cases} \tag{8}$$

Now, from (7) and (8) yield

$$\begin{cases} \|N_1(\varphi,\phi)\|^2 \leq \big[c_{emb}\|r_1\| + ac_{emb}^{\delta_1+1}(mes(\Omega))^{\frac{1-\delta_1}{2}}\|\varphi\|^{\delta_1} + bc_{emb}^{\delta_1+1}(mes(\Omega))^{\frac{1-\delta_1}{2}}\|\phi\|^{\delta_1}\big]^2, \\ \|N_2(\varphi,\phi)\|^2 \leq \big[c_{emb}\|r_2\| + cc_{emb}^{\delta_2+1}(mes(\Omega))^{\frac{1-\delta_2}{2}}\|\varphi\|^{\delta_2} + dc_{emb}^{\delta_2+1}(mes(\Omega))^{\frac{1-\delta_2}{2}}\|\phi\|^{\delta_2}\big]^2. \end{cases}$$

If we put (just a notation)
$$\begin{cases} t = ac_{emb}^{\delta_1+1}(mes(\Omega))^{\frac{1-\delta_1}{2}}, \\ l = bc_{emb}^{\delta_1+1}(mes(\Omega))^{\frac{1-\delta_1}{2}}, \\ j = cc_{emb}^{\delta_2+1}(mes(\Omega))^{\frac{1-\delta_2}{2}}, \\ h = dc_{emb}^{\delta_2+1}(mes(\Omega))^{\frac{1-\delta_2}{2}}, \end{cases}$$

then,
$$\begin{cases} \|N_1(\varphi,\phi)\|^2 \leq \big[c_{emb}\|r_1\| + max(t,l)(\|\varphi\|^{\delta_1} + \|\phi\|^{\delta_1})\big]^2, \\ \|N_2(\varphi,\phi)\|^2 \leq \big[c_{emb}\|r_2\| + max(j,h)(\|\varphi\|^{\delta_2} + \|\phi\|^{\delta_2})\big]^2. \end{cases}$$

Thus,

$$\begin{cases} \|N_1(\varphi,\phi)\|^2 \leq 2c_{emb}^2\|r_1\|^2 + 4max^2(t,l)(\|\varphi\|^{2\delta_1} + \|\phi\|^{2\delta_1}), \\ \|N_2(\varphi,\phi)\|^2 \leq 2c_{emb}^2\|r_2\|^2 + 4max^2(j,h)(\|\varphi\|^{2\delta_2} + \|\phi\|^{2\delta_2}). \end{cases} \tag{9}$$

By adding the two inequality in (9) , we have
$$\|N(\varphi,\phi)\|^2 \leq \underbrace{2c_{emb}^2\|r\|^2}_{=C} + \underbrace{4(max^2(t,l) + max^2(j,h))}_{=D} max(\|(\varphi,\phi)\|^{2\delta_1}, \|(\varphi,\phi)\|^{2\delta_2}). \tag{10}$$

It follows that for any $(\varphi,\phi) \in B(0;R) \subset U$,

$$\|N(\varphi,\phi)\| \leq R, \quad \text{with} \quad \sqrt{C + D\, max(R^{2\delta_1}, R^{2\delta_2})} < R.$$

Hence, if R is large enough, then N maps $B(0;R)$ into itself. □

Now, we can prove our main result, which is Theorem 2.2.

Proof. (Theorem 2.2) To prove Theorem 2.2, we can apply the Schauder fixed point theorem. It follows from Lemmas 4.1, 4.2 and 4.3 that there is at least one fixed point $(\varphi, \phi) \in U$ of N (which mean the system (1) have a weak solution in U). This completes the proof. $\square$

5 Particular Case

There are p and k Lipschitz continuous functions with respect to the second variable, i.e., there are constants $c_1, c_2 \in \mathbb{R}^+$ for almost every $x \in \Omega$ and for any $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2$,

$$\begin{cases} \|p(z, x_1) - p(z, x_2)\|_{L^2(\Omega)} \leq c_1\|x_1 - x_2\|_{L^2(\Omega)\times L^2(\Omega)}, \\ \|k(z, y_1) - k(z, y_2)\|_{L^2(\Omega)} \leq c_2\|y_1 - y_2\|_{L^2(\Omega)\times L^2(\Omega)}. \end{cases} \tag{11}$$

The contraction principle is applied to have the following result.

Theorem 5.1. *Let the Carathéodory functions p, k be Lipschitzian continuous with respect to the second variable with constants* $c_i > 0$ $(i = 1, 2)$ *such that* $|c| < c_{emb}^{-2}$ $(c = (c_1, c_2))$. *Then, there is a unique fixed point* $(\varphi, \phi) \in U$.

To prove Theorem 5.1 we need the following contraction principle.

Theorem 5.2. (*Contraction principle, see* [1]) *Let* ψ *be a contraction mapping from* X *to* X. *Then* ψ *admits a unique fixed-point in* X.

So to use the contraction principle we must prove that the operator N is contraction.

Lemma 5.1. *The* N *operator is a contraction.*

Proof. For any $(\varphi, \phi) \in U$ we also have $(\varphi, \phi) \in V$, then $(p(z, \varphi, \phi), k(z, \varphi, \phi)) \in (L^2(\Omega))^2$. Then, for all $(\varphi_1, \phi_1), (\varphi_2, \phi_2) \in U$, we have

$$\begin{cases} \|N_1(\varphi_1, \phi_1) - N_1(\varphi_2, \phi_2)\|_{E^{s,2}} = \sup_{\|\mu\|\leq 1}|\langle N_1(\varphi_1, \phi_1) - N_1(\varphi_2, \phi_2), \mu\rangle|, \\ \|N_2(\varphi_1, \phi_1) - N_2(\varphi_2, \phi_2)\|_{E^{s,2}} = \sup_{\|\nu\|\leq 1}|\langle N_2(\varphi_1, \phi_1) - N_2(\varphi_2, \phi_2)\rangle, \nu)|. \end{cases}$$

This means that,

$$\begin{cases} \|N_1(\varphi_1, \phi_1) - N_1(\varphi_2, \phi_2)\|_{E^{s,2}} = \sup_{\|\mu\|\leq 1}|\int_\Omega [p(z, \varphi_1, \phi_1) - p(z, \varphi_2, \phi_2)]\mu(z)dz|, \\ \|N_2(\varphi_1, \phi_1) - N_2(\varphi_2, \phi_2)\|_{E^{s,2}} = \sup_{\|\nu\|\leq 1}|\int_\Omega [k(z, \varphi_1, \phi_1) - k(z, \varphi_2, \phi_2)]\nu(z)dz|. \end{cases}$$

Using the hypothesis (11), we get

$$\begin{cases} \|N_1(\varphi_1, \phi_1) - N_1(\varphi_2, \phi_2)\|^2_{E^{s,2}} \leq c_1^2 c_{emb}^4\|(\varphi_1, \phi_1) - (\varphi_2, \phi_2)\|^2_U, \\ \|N_2(\varphi_1, \phi_1) - N_2(\varphi_2, \phi_2)\|^2_{E^{s,2}} \leq c_2^2 c_{emb}^4\|(\varphi_1, \phi_1) - (\varphi_2, \phi_2)\|^2_U. \end{cases}$$

Consequently, N is a contraction if $c_{emb}^2|c| < 1$. $\square$

Now, we can prove Theorem 5.1.

Proof. We have from Lemma 5.1 that if $c_{emb}^2|c| < 1$ then N is a contraction, so we can apply the contraction principle to get that there is a unique fixed point $(\varphi, \phi) \in U$ of the operator N. That is, (φ, ϕ) is the unique weak solution of (1). □

6 Conclusion

In conclusion, our article allowed us to answer the question of having a solution to strongly nonlinear elliptic system in the case of fractional derivatives, this result is new and can be deemed as a fractional version of the classical theorems. Lastly, this study can extend to more general boundary value systems involving fractional derivatives such as systems of convection-diffusion-reaction and find the appropriate numerical methods. We can also try to find an application to these models in image processing.

Acknowledgement The authors would like to express their gratitude to the referees both for their careful assessment and for their fruitful comments regarding the first version of this paper. The author S. Dob highly thankful and grateful to the laboratory of Applied Mathematics and History and Didactics of Mathematics for supporting and facilitating this research work.

Conflicts of Interest The authors declare that there is no conflict of interest in this article.

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